Applied Mathematics and Computation 445 (2023) 127842

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

# Spanning trees with at most $k$ leaves in 2-connected $K_{1,r}$-free graphs

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ARTICLE INFO

*Article history:*
Received 30 May 2022
Revised 18 November 2022
Accepted 3 January 2023

*MSC:*
05C05
05C07
05C69

*Keywords:*
Spanning tree
Leaf
Independence number
$K_{1,r}$-free

ABSTRACT

A vertex with degree one and a vertex with degree at least three are called a leaf and a branch vertex in a tree, respectively. In this paper, we obtain that every 2-connected $K_{1,r}$-free graph $G$ contains a spanning tree with at most $k$ leaves if $\alpha(G) \le k + \lceil \frac{k+1}{r-3} \rceil - \lfloor \frac{1}{|r-k-3|+1} \rfloor$, where $k \ge 2$ and $r \ge 4$. The upper bound is best possible. Furthermore, we prove that if a connected $K_{1,4}$-free graph $G$ satisfies that $\alpha(G) \le 2k+5$, then $G$ contains either a spanning tree with at most $k$ branch vertices or a block $B$ with $\alpha(B) \le 2$. A related conjecture for 2-connected claw-free graphs is also posed.



## 1. Introduction

In this paper, we only consider simple and undirected graphs. Let $G$ be a graph and $v \in V(G)$. We denote the degree of $v$ by $deg_G(v)$ and the vertices which are adjacent to $v$ by $N_G(v)$. For a set $S \subseteq V(G)$, the subgraph induced by $S$ and $V(G) \setminus S$ are denoted by $G[S]$ and $G - S$, respectively. We denote the number of vertices in $S$ by $|S|$.

A subset $X$ is *independent* in $G$ if $G[X]$ has no edge. The *independence number* of $G$ is denoted by $\alpha(G)$, which means the maximum number of vertices in an independent set of $G$. Define $\sigma_k(G) = \min\{\sum_{x \in X} deg_G(x) \mid X$ is independent in $G$ and $|X| = k\}$. $G$ is called *$K_{1,r}$-free* if $K_{1,r}$ is not an induced subgraph of $G$. We write claw-free graph for the $K_{1,3}$-free graph. The *center of a claw* refers to the vertex of degree 3 in $K_{1,3}$ and $x$-claw refers to a claw with center $x$.

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[1] This author's work was partially supported by NSFC (No. 11871239).
[2] This author's work was partially supported by NSFC (No. 12201472).
[3] This author's work was partially supported by NSFC ( No. 11971196).
[4] This author's work was partially supported by NSFC (No. 11971158), the Youth Project Funds of Hubei Provincial Department of Education (No. 202010401301002) and the General Project Funds of Hubei Provincial Department of Science and Technology (No. 2021CFB476).

https://doi.org/10.1016/j.amc.2023.127842
0096-3003/

We call $v$ a *leaf* of tree $T$ if $deg_T(v) = 1$ and denote $L(T)$ the set of leaves of $T$. A vertex $v$ with $deg_T(v) \geq 3$ is called a *branch vertex* of tree $T$ and define $B(T)$ the set of branch vertices of $T$.

There are some well-known results such as Ore's Theorem [8] and Chvátal–Erdős's Theorem [4] related to conditions of degree sum and independence number ensuring a Hamiltonian path in $G$, respectively. Note that a Hamiltonian path is a spanning tree with two leaves. With this viewpoint, researchers gave several results concerning about such two types of conditions to guarantee the existence of spanning tree with bounded leaves(see the survey paper [9]).

The following two results generalize Ore's Theorem [8] and Chvátal–Erdős's Theorem [4], respectively.

**Theorem 1.1** (Broerma and Tuinstra [1]). *Let $k \geq 2$. If $G$ is a connected graph of order $n$ such that $\sigma_2(G) \geq n-k+1$, then $G$ has a spanning tree with at most $k$ leaves.*

**Theorem 1.2** (Win [10]). *Let $k \geq 2$. If $G$ is an $m$-connected graph such that $\alpha(G) \leq m+k-1$, then $G$ has a spanning tree with at most $k$ leaves.*

Since there are many researches on Hamiltonian path problem in $K_{1,r}$-free graphs, it is also natural for us to search for conditions for $K_{1,r}$-free graphs to ensure the existence of spanning trees with bounded leaves. Here are some related results on $K_{1,r}$-free graphs.

**Theorem 1.3** (Kano et al. [6]). *Let $k \geq 2$. If $G$ is a connected claw-free graph of order $n$ such that $\sigma_{k+1}(G) \geq n-k$, then $G$ has a spanning tree with at most $k$ leaves.*

**Theorem 1.4** (Kyaw [7]). *Let $G$ be a connected $K_{1,4}$-free graph of order $n$.*

(i) *If $\sigma_3(G) \geq n$, then $G$ has a Hamiltonian path.*
(ii) *If $\sigma_{k+1}(G) \geq n - \frac{k}{2}$ for some integer $k \geq 3$, then $G$ has a spanning tree with at most $k$ leaves.*

**Theorem 1.5** (Chen et al. [2]). *Let $m \geq 2$. If $G$ is an $m$-connected $K_{1,4}$-free graph of order $n$ such that $\sigma_{m+3}(G) \geq n+2m-2$, then $G$ has a spanning tree with at most 3 leaves.*

**Theorem 1.6** (Chen et al. [3]). *If $G$ is a connected $K_{1,5}$-free graph of order $n$ such that $\sigma_5(G) \geq n-1$, then $G$ has a spanning tree with at most 4 leaves.*

**Theorem 1.7** (Hu and Sun [5]). *If $G$ is a connected $K_{1,5}$-free graph of order $n$ such that $\sigma_6(G) \geq n-1$, then $G$ has a spanning tree with at most 5 leaves.*

In this paper, we consider $\alpha(G)$ for a 2-connected $K_{1,r}$-free graph with $r \geq 4$ to guarantee the existence of a spanning tree with bounded leaves.

**Theorem 1.8.** *Let $k \geq 2$ and $r \geq 4$. If $G$ is a 2-connected $K_{1,r}$-free graph such that $\alpha(G) \leq k + \lceil \frac{k+1}{r-3} \rceil - \lfloor \frac{1}{|r-k-3|+1} \rfloor$, then $G$ has a spanning tree with at most $k$ leaves.*

By taking $r = 4$ in Theorem 1.8, we have the following corollary.

**Corollary 1.9.** *Let $k \geq 2$. If $G$ is a 2-connected $K_{1,4}$-free graph such that $\alpha(G) \leq 2k+1$, then $G$ has a spanning tree with at most $k$ leaves.*

Note that a tree with at most $k$ leaves contains at most $k-2$ branch vertices. We can easily obtain the following corollary.

**Corollary 1.10.** *Let $k \geq 0$. If $G$ is a 2-connected $K_{1,4}$-free graph such that $\alpha(G) \leq 2k+5$, then $G$ has a spanning tree with at most $k$ branch vertices.*

With the same independence number condition of Corollary 1.10, we further provide the following result for connected $K_{1,4}$-free graphs.

**Theorem 1.11.** *Let $k \geq 0$. If $G$ is a connected $K_{1,4}$-free graph such that $\alpha(G) \leq 2k+5$, then one of the following two statements holds:*

(i) *$G$ has a spanning tree with at most $k$ branch vertices;*
(ii) *there exists a block $B$ in $G$ with $\alpha(B) \leq 2$.*

We provide the following conjecture for connected claw-free graphs to end this section.

**Conjecture 1.12.** *Let $k \geq 2$. If $G$ is a 2-connected claw-free graph such that $\alpha(G) \leq 2k+2$, then $G$ has a spanning tree with at most $k$ leaves.*

In next section, we show that the upper bounds of $\alpha(G)$ are sharp in Theorem 1.8 and Conjecture 1.12 if it is true. We prove Theorem 1.8 and Theorem 1.11 in Sections 3 and 4, respectively.

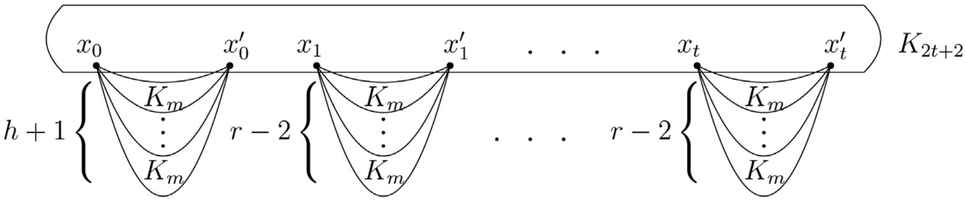


**Fig. 1.** Graph $G_1$.

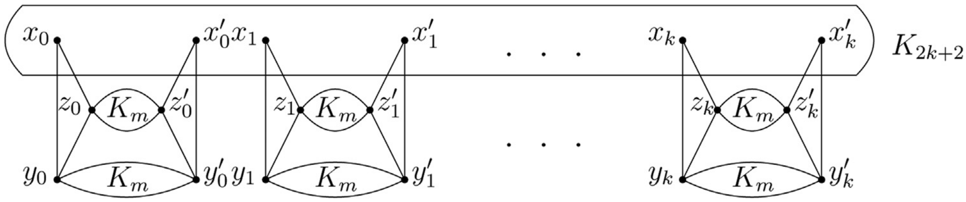


**Fig. 2.** Graph $G_3$.

## 2. Sharpness of Theorem 1.8 and Conjecture 1.12

First, we show that the upper bound of $\alpha(G)$ in Theorem 1.8 is sharp. This is shown in the following examples $G_1$ and $G_2$.

Denote $t = \lfloor \frac{k+1}{r-3} \rfloor$ and $h = k+1-t(r-3)$.

**Case 1.** $r \neq k+3$.

In this case, $\lfloor \frac{1}{|r-k-3|+1} \rfloor = 0$.

If $h \neq 0$, we construct a graph $G_1$ from a complete graph $K_{2t+2}$ with $V(K_{2t+2}) = \{x_0, x'_0, x_1, x'_1, \ldots, x_t, x'_t\}$ and $(r-2)t + h+1$ complete graphs $K_m (m \geq 3)$ by identifying $r-2$ complete graphs $K_m$ with every pair of $\{x_i, x'_i\}$ for $1 \leq i \leq t$ and by identifying $h+1$ complete graphs $K_m$ with $\{x_0, x'_0\}$ (see Fig. 1). Then $G_1$ is 2-connected $K_{1,r}$-free and $\alpha(G_1) = t(r-2) + h+1 = t(r-2) + k+1-t(r-3)+1 = k+1+t+1 = k+1+\lceil \frac{k+1}{r-3} \rceil$. However, for every spanning tree $T_1$ of $G_1$, we have $|L(T_1)| \geq t(r-3)+h = k+1$. **Case 2.** $r = k+3$.

In this case, $\lceil \frac{k+1}{r-3} \rceil = 2$ and $\lfloor \frac{1}{|r-k-3|+1} \rfloor = 1$.

We construct a graph $G_2$ from a pair of vertex set $\{x_0, x'_0\}$ and $r-1$ complete graphs $K_m$ $(m \geq 3)$ by identifying $r-1$ complete graphs $K_m$ with $\{x_0, x'_0\}$. Then $G_2$ is 2-connected $K_{1,r}$-free and $\alpha(G_2) = r-1 = k+2$, but $G_2$ has no spanning tree with at most $k$ leaves.

Next, we show that the upper bound $2k+2$ in Conjecture 1.12 is sharp if it is true. For $0 \leq i \leq k$, let $T_i$ and $T'_i$ be two triangles with $V(T_i) = \{x_i, y_i, z_i\}$ and $V(T'_i) = \{x'_i, y'_i, z'_i\}$, respectively. Consider a graph $G_3$ constructed from a complete graph $K_{2k+2}$ with $V(K_{2k+2}) = \{x_0, x'_0, x_1, x'_1, \ldots, x_k, x'_k\}$ and $2k+2$ complete graphs $K_m$ $(m \geq 3)$ by identifying $2k+2$ complete graphs $K_m$ with every pair of $\{y_i, y'_i\}$ and $\{z_i, z'_i\}$ for $0 \leq i \leq k$, respectively (see Fig. 2). Then $G_3$ is 2-connected claw-free with $\alpha(G_3) = 2k+3$, but $G_3$ has no spanning tree with at most $k$ leaves.

## 3. Proof of Theorem 1.8

We begin with some additional notations. Let $x$ and $y$ be two vertices of $G$, we denote the distance between $x$ and $y$ in $G$ by $d_G(x,y)$. Let $u$ and $v$ be two vertices in a spanning tree $T$ of $G$, the unique path from $u$ to $v$ in $T$ is denoted by $T[u,v]$. We write $T[u,v]-\{u,v\}$, $T[u,v]-\{u\}$, $T[u,v]-\{v\}$ by $T(u,v)$, $T(u,v]$ and $T[u,v)$, respectively. Set $I(T) = V(T) - L(T)$ and $f(T) = \max_{v \in L(T)} f(T,v)$, where $f(T,v) = \sum_{z \in I(T)} (deg_T(z)-2)d_T(v,z)$. Note that $(deg_T(z)-2)d_T(v,z) = 0$ if $deg_T(z) = 2$. Set $g(T) = \sum_{x \in L(T)} g(T,x)$, where $g(T,x) = \max\{d_T(x,y) | y \in N_G(x)\}$.

**Proof of Theorem 1.8..** Suppose that $G$ is a 2-connected $K_{1,r}$-free graph and every spanning tree has at least $k+1$ leaves in $G$. We choose a spanning tree $T$ of $G$ satisfying that

- $(C1)$ $|L(T)|$ is as small as possible;
- $(C2)$ Subject to $(C1)$, $f(T)$ is as large as possible;
- $(C3)$ Subject to $(C1)$ and $(C2)$, $g(T)$ is as large as possible.

Assume that $L(T) = \{x_0, x_1, \ldots, x_t\}$ and $f(T) = f(T, x_0)$. Then $t \geq k$. $T$ is considered as a rooted tree and $x_0$ is the root of $T$. For $1 \leq i \leq t$, $r_i$ is the last branch vertex of $T$ on $T[x_0, x_i]$ and $r_i^+$ is the successor of $r_i$ on $T[x_0, x_i]$. For $v \in V(T) - \{x_0\}$, the predecessor of $v$ is denoted by $v^-$ on $T[x_0, v]$. □

**Claim 3.1.** *$L(T)$ is independent in $G$.*

**Proof.** Assume that $x_ix_j \in E(G)$ for some $i$ and $j$ with $0 \le i \ne j \le t$. Then $T^* = T - \{r_ir_i^+\} + \{x_ix_j\}$ is a spanning tree with $L(T^*) = (L(T) - \{x_i, x_j\}) \cup \{r_i^+\}$, contradicting $(C1)$. This proves Claim 3.1. □

**Remark 3.1.** From the proof of Claim 3.1, we know that for every spanning tree $T^*$ of $G$ with $|L(T^*)| \le |L(T)|$, then $L(T^*)$ is independent in $G$ with $|L(T^*)| = |L(T)|$.

**Claim 3.2.** *For $1 \le i \le t$, there is no neighbour of $x_0$ on $T(r_i, x_i)$.*

**Proof.** Assume that $y \in N_G(x_0)$ with $y \in V(T(r_i, x_i))$ for some $1 \le i \le t$. Then $T^* = T - \{yy^-\} + \{x_0y\}$ is a spanning tree of $G$. If $y^- = r_i$, then $|L(T^*)| < |L(T)|$, contrary to $(C1)$; if $y^- \ne r_i$, then $T^*$ satisfies $(C1)$. Note that $B(T^*) = B(T)$. Then $d_{T^*}(z, x_i) = d_{T^*}(x_0, x_i) + d_T(z, x_0)$ for any $z \in B(T)$. Since $d_{T^*}(x_0, x_i) > 1$, we have $d_{T^*}(z, x_i) > d_T(z, x_0)$. Thus $f(T^*, x_i) > f(T, x_0)$. Then we have $f(T^*) > f(T)$, contrary to $(C2)$. □

For $1 \le i_1 < \ldots < i_l \le t$, denote by $r_{i_1 \ldots i_l}$ the last common vertex of the paths $T[x_0, x_{i_1}], \ldots, T[x_0, x_{i_l}]$. We denote the successor of $r_{ij}$ on $T[r_{ij}, x_i]$ and $T[r_{ij}, x_j]$ by $r_{ij}^+$ and $r_{ji}^+$, respectively. Denote the predecessor of $r_{ij}$ on $T[x_0, r_{ij}]$ by $r_{ij}^-$. The predecessor of $y$ on $T(r_{ij}, x_j)$ is denoted by $y^-$.

**Claim 3.3.** *$N_G(x_i) \subseteq V(T(x_0, x_i))$ for $1 \le i \le t$.*

**Proof.** Assume that there exists $x_j \in L(T) - \{x_0, x_i\}$ satisfying that $x_i$ has a neighbour $y$ on $T(r_{ij}, x_j)$. Obviously, $r_i \in V(T[r_{ij}, x_i))$ and $r_j \in V(T[r_{ij}, x_j))$.

Set $T^* = T - \{r_ir_i^+\} + \{x_iy\}$. Then $T^*$ is a spanning tree with $L(T^*) = (L(T) - \{x_i\}) \cup \{r_i^+\}$. Then $I(T^*) = (I(T) - \{r_i^+\}) \cup \{x_i\}$. Note that $d_{T^*}(x_0, r_i) = d_T(x_0, r_i)$, $deg_{T^*}(r_i) = deg_T(r_i) - 1$, $d_{T^*}(x_0, y) = d_T(x_0, y)$ and $deg_{T^*}(y) = deg_T(y) + 1$. Note that $deg_{T^*}(x_i) = 2$, $deg_T(r_i^+)$=2 and $(deg_T(z) - 2)d_T(x_0, z) = (deg_{T^*}(z) - 2)d_{T^*}(x_0, z)$ for all $z \in I(T^*) \cap I(T) - \{r_i, y\}$. Hence,

$$\begin{aligned} f(T^*, x_0) - f(T, x_0) &= \sum_{z \in I(T^*)} (deg_{T^*}(z) - 2)d_{T^*}(x_0, z) - \sum_{z \in I(T)} (deg_T(z) - 2)d_T(x_0, z) \\ &= \sum_{z \in I(T^*) \setminus \{x_i\}} (deg_{T^*}(z) - 2)d_{T^*}(x_0, z) - \sum_{z \in I(T) \setminus \{r_i^+\}} (deg_T(z) - 2)d_T(x_0, z) \\ &= \sum_{z \in \{r_i, y\}} (deg_{T^*}(z) - 2)d_{T^*}(x_0, z) - \sum_{z \in \{r_i, y\}} (deg_T(z) - 2)d_T(x_0, z) \\ &= \sum_{z \in \{r_i, y\}} (deg_{T^*}(z) - deg_T(z))d_T(x_0, z) \\ &= d_T(x_0, y) - d_T(x_0, r_i). \end{aligned}$$

This together with $(C2)$ implies that $d_T(x_0, r_i) \ge d_T(x_0, y)$. (1)

If $y \in V(T(r_{ij}, r_j])$, we set $T' = T - \{yy^-\} + \{x_iy\}$. Then $T'$ is a spanning tree and $I(T') = (I(T) - \{y^-\}) \cup \{x_i\}$. If $deg_T(y^-) \ge 3$, we have $L(T') = L(T) - \{x_i\}$, contradicting $(C1)$. So $deg_T(y^-) = 2$. Note that $(deg_T(z) - 2)d_T(x_0, z) = (deg_{T'}(z) - 2)d_{T'}(x_0, z)$ for all $z \in I(T^*) \cap I(T) - V(T[y, r_j])$. We have

$$\begin{aligned} f\left(T', x_0\right) - f(T, x_0) &= \sum_{z \in I(T')} (deg_{T'}(z) - 2)d_{T'}(x_0, z) - \sum_{z \in I(T)} (deg_T(z) - 2)d_T(x_0, z) \\ &= \sum_{z \in I(T') \setminus \{x_i\}} (deg_{T'}(z) - 2)d_{T'}(x_0, z) - \sum_{z \in I(T) \setminus \{y^-\}} (deg_T(z) - 2)d_T(x_0, z) \\ &= \sum_{z \in V\left(T\left[y, r_j\right]\right)} (deg_{T'}(z) - 2)d_{T'}(x_0, z) - \sum_{z \in V\left(T\left[y, r_j\right]\right)} (deg_T(z) - 2)d_T(x_0, z) \\ &= \sum_{z \in V\left(T\left[y, r_j\right]\right)} (deg_T(z) - 2)(d_{T'}(x_0, z) - d_T(x_0, z)) \\ &\ge \sum_{z \in V\left(T\left[y, r_j\right]\right)} (deg_T(z) - 2)[d_T(x_0, r_i) - d_T(x_0, y) + 2]. \end{aligned}$$

This together with (1) implies that $f(T', x_0) - f(T, x_0) \ge 2 \sum_{z \in V(T[y, r_j])} (deg_T(z) - 2)$. Noting that $r_j \in V(T[y, r_j])$, we get $f(T') - f(T) \ge f(T', x_0) - f(T, x_0) \ge 2[deg_T(r_j) - 2] \ge 2$, contrary to $(C2)$.

If $y \in V(T(r_j, x_j))$, we set $T'' = T - \{r_jr_j^+\} + \{x_iy\}$. Then $T''$ is a spanning tree and $I(T'') = (I(T) - \{r_j^+\}) \cup \{x_i\}$. If $y^- = r_j$, then $L(T'') = L(T) - \{x_i\}$, contrary to $(C1)$. Thus, $y^- \ne r_j$. Note that $deg_{T''}(r_j) = deg_T(r_j) - 1$, $d_{T''}(x_0, r_j) = d_T(x_0, r_j)$,

$deg_T(y) = 2$, $deg_{T''}(y) = deg_T(y) + 1 = 3$. From (1), we have $d_{T''}(x_0, y) \geq d_T(x_0, r_i) + 2 \geq d_T(x_0, y) + 2$. By the similar discussion to that in the proof of (1),

$$\begin{aligned} f\left(T'\prime, x_0\right) - f(T, x_0) &= \sum_{z\in I(T'\prime)} (deg_{T'\prime}(z) - 2)d_{T'\prime}(x_0, z) - \sum_{z\in I(T)} (deg_T(z) - 2)d_T(x_0, z) \\ &= \sum_{z\in I(T'\prime)\setminus\{x_i\}} (deg_{T'\prime}(z) - 2)d_{T'\prime}(x_0, z) - \sum_{z\in I(T)\setminus\{r_j^+\}} (deg_T(z) - 2)d_T(x_0, z) \\ &= \sum_{z\in\{r_j, y\}} (deg_{T'\prime}(z) - 2)d_{T'\prime}(x_0, z) - \sum_{z\in\{r_j, y\}} (deg_T(z) - 2)d_T(x_0, z) \\ &= \left(deg_{T'\prime}\left(r_j\right) - deg_T\left(r_j\right)\right)d_T\left(x_0, r_j\right) + d_{T'\prime}(x_0, y) \\ &= d_{T'\prime}(x_0, y) - d_T\left(x_0, r_j\right) \\ &> d_{T'\prime}(x_0, y) - d_T(x_0, y). \\ &\geq 2. \end{aligned}$$

This implies that $f(T'') - f(T) \geq f(T'', x_0) - f(T, x_0) > 2$, also contradicting $(C2)$. This proves Claim 3.3. □

**Claim 3.4.** *Let $1 \leq i \neq j \leq t$. Then $r_{ij}^- \notin N_G(x_i)$ and $r_{ij}^+ \notin N_G(x_0)$.*

**Proof.** Suppose that Claim 3.4 is false. Set

$$T^* = \begin{cases} T - \left\{r_{ij}^- r_{ij}\right\} + \left\{x_i r_{ij}^-\right\}, & \text{if } x_i r_{ij}^- \in E(G) \\ T - \left\{r_{ij} r_{ij}^+\right\} + \left\{x_0 r_{ij}^+\right\}, & \text{if } x_0 r_{ij}^+ \in E(G). \end{cases}$$

Then $T^*$ is a spanning tree with $|L(T^*)| < |L(T)|$, contrary to $(C1)$. □

For $0 \leq i \leq t$, let $y_i$ be the neighbour of $x_i$ such that $d_T(x_i, y_i) = g(T, x_i)$. According to Claim 3.3, $y_i \in V(T(x_0, x_i))$ for $1 \leq i \leq t$. We denote the successor of $y_i$ on $T[x_0, x_i]$ by $y_i^+$. Set $I_1 = \{i \in [1, t] : y_i \in V(T[r_i, x_i))\}$ and $I_2 = \{i \in [1, t] : y_i \in V(T(x_0, r_i))\}$. Obviously, $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = [1, t]$.

**Claim 3.5.** *For $i \in I_1$, there exists $z_i \in V(T[y_i, x_i))$ satisfying that $z_i^+ \notin N_G[x_i]$ and $V(T[y_i, z_i]) \subseteq N_G(x_i)$ where $N_G[x_i] = N_G(x_i) \cup \{x_i\}$ and $z_i^+$ is the successor of $z_i$ on $T[x_0, x_i]$.*

**Proof.** Suppose that Claim 3.5 is false. Then there is an integer $i \in I_1$ such that $N_G[x_i] \cap V(T[r_i, x_i]) = V(T[y_i, x_i])$. Since $G$ is 2-connected, $G - y_i$ is connected. There is $u_i \in V(T(y_i, x_i))$ such that $u_i$ has a neighbour $v_i$ in $T - T[y_i, x_i]$. Set $T^* = T - \{u_i^- u_i\} + \{u_i^- x_i\}$. Then $T^*$ is a spanning tree with $L(T^*) = (L(T) - \{x_i\}) \cup \{u_i\}$ that satisfies $(C1)$ and $(C2)$. Noting that $d_{T^*}(u_i, y_i) = d_T(x_i, y_i)$, we have $d_{T^*}(u_i, v_i) > d_T(x_i, y_i)$, which implies that $g(T^*, u_i) > g(T, x_i)$. On the other hand, by Claims 3.2 and 3.3, we have $N_G(x_j) \cap V(T(r_i, x_i)) = \emptyset$ for $0 \leq j \neq i \leq t$. Hence, $g(T^*, x_j) = g(T, x_j)$ for $0 \leq j \neq i \leq t$. We have $g(T^*) > g(T)$, contrary to $(C3)$. □

By Claim 3.5, there exists $z_i \in V(T[y_i, x_i))$ satisfying that $z_i^+ \notin N_G[x_i]$, $V(T[y_i, z_i]) \subseteq N_G(x_i)$ and let $L_1'(T) = \{z_i : i \in I_1\}$. Denote $z_j = y_j$ for $j \in I_2$ and let $L_2'(T) = \{z_j : j \in I_2\}$. For $h = 1, 2$, define $X^h = \{x_i : i \in I_h\}$ and $L_h(T) = \{z_i^+ : z_i \in L_h'(T)\}$.

By the choice of $z_i$ for $i \in I_1 \cup I_2$, we define two surjections $\theta_h : X^h \to L_h'(T)$ for $h = 1, 2$. Note that $z_i \in V(T(y_i, x_i))$ for $i \in I_1$. Since $V(T(y_i, x_i)) \cap V(T(y_j, x_j)) = \emptyset$ for $i \neq j \in I_1$, $\theta_1$ is a bijection. Thus $|L_1(T)| = |L_1'(T)| = |I_1|$.

**Claim 3.6.** $|L_2(T)| \geq |L_2'(T)| \geq \lceil \frac{|I_2|}{r-3} \rceil$.

**Proof.** Let $\theta_2^{-1}(z_i)$ be the preimage of $z_i$ in $X^2$ for $z_i \in L_2'(T)$. Suppose that $\theta_2^{-1}(z_i) = \{x_{i_s} : s \geq 1\}$ for $i \in I_2$ and $z_i = z_{i_1} = \ldots = z_{i_s}$. By Claim 3.3, we have $z_i \in V(T(x_0, x_{i_j}))$ for $1 \leq j \leq s$. Hence, $z_i \in V(T(x_0, r_{i_1 \ldots i_s}])$. We claim that

$$\{z_i^-, x_{i_1}, \ldots, x_{i_s}\} \cup \{z_{i_j}^+ : 1 \leq j \leq s\} \text{ is independent in } G. \qquad (*)$$

Suppose to the contrary that $(*)$ is false. By Claim 3.1, $\{x_{i_1}, \ldots, x_{i_s}\}$ is independent. Then one of the following cases occurs.

- $z_i^- z_{i_j}^+ \in E(G)$ for some $j \in [1, s]$. If $deg_T(z_i) \geq 3$, then $T^* = T - \{z_i^- z_i, z_i z_{i_j}^+\} + \{z_i x_{i_j}, z_i^- z_{i_j}^+\}$ is a spanning tree with $L(T^*) = L(T) - \{x_{i_j}\}$, contrary to $(C1)$. Hence, $deg_T(z_i) = 2$. For any $h \in [1, s] \setminus \{j\}$, $T^{(1)} = T - \{z_i^- z_i, z_i z_{i_j}^+, r_{i_h} r_{i_h}^+\} + \{z_i x_{i_j}, z_i^- z_{i_j}^+, z_i x_{i_h}\}$ is a spanning tree with $L(T^{(1)}) = (L(T) - \{x_{i_j}, x_{i_h}\}) \cup \{r_{i_h}^+\}$, contrary to $(C1)$.
- $z_i^- x_{i_j} \in E(G)$ for some $j \in [1, s]$. It follows that $d_T(x_{i_j}, z_i^-) = d_T(x_{i_j}, z_i) + 1 > g(T, x_{i_j})$, contrary to the choice of $z_i$.
- $x_{i_j} z_{i_h}^+ \in E(G)$ for some $j \neq h \in [1, s]$. If $z_{i_j}^+ = z_{i_h}^+$, then $T^{(2)} = T - \{z_i z_{i_j}^+, x_{i_j}^- x_{i_j}\} + \{z_i x_{i_j}, z_{i_j}^+ x_{i_j}\}$ is a spanning tree with $L(T^{(2)}) \subseteq (L(T) - \{x_{i_j}\}) \cup \{x_{i_j}^-\}$. It is straight to check that $f(T^{(2)}, x_0) > f(T, x_0)$, which indicates that $f(T^{(2)}) > f(T)$, contrary to $(C2)$. Hence $z_{i_j}^+ \neq z_{i_h}^+$. Then $T^{(3)} = T - \{z_i z_{i_h}^+\} + \{x_{i_j} z_{i_h}^+\}$ is a spanning tree with $L(T^{(3)}) = L(T) - \{x_{i_j}\}$, contrary to $(C1)$.
- $z_{i_j}^+ z_{i_h}^+ \in E(G)$ for some $j \neq h \in [1, s]$. Then $T^{(4)} = T - \{z_i z_{i_j}^+, z_i z_{i_h}^+\} + \{z_{i_j}^+ z_{i_h}^+, z_i x_{i_j}\}$ is a spanning tree with $L(T^{(4)}) = L(T) - \{x_{i_j}\}$, contrary to $(C1)$.

Therefore, $(*)$ is true. Since $G$ is $K_{1,r}$-free and $z$ is adjacent to each vertex in $\{z_i^-, x_{i_1}, \ldots, x_{i_s}\} \cup \{z_{i_j}^+ : 1 \le j \le s\}$, we have $s \le r-3$. This implies that $|L_2(T)| \ge |L_2'(T)| \ge \lceil \frac{|I_2|}{r-3} \rceil$. □

Set $U = L(T) \cup L_1(T) \cup L_2(T)$. By the definitions of $L(T)$, $L_1(T)$ and $L_2(T)$, three vertex sets $L(T)$, $L_1(T)$ and $L_2(T)$ are disjoint. Thus $|U| = |L(T)| + |L_1(T)| + |L_2(T)|$.

**Claim 3.7.** *$U$ is independent in $G$.*

**Proof.** First, we show that $L_1(T) \cup L_2(T)$ is independent. Set $T_a = T - \{z_i z_i^+, z_j z_j^+\} + \{z_i x_i, z_j x_j\}$ for $z_i \ne z_j \in L_1'(T) \cup L_2'(T)$. By Claim 3.3, $z_i \in V(T(x_0, x_i))$ and $z_j \in V(T(x_0, x_j))$. Then $T_a$ is a spanning tree with $L(T_a) \subseteq (L(T) - \{x_i, x_j\}) \cup \{z_i^+, z_j^+\}$. By Remark 3.1, $L(T_a)$ is independent in $G$. Hence, $z_i^+ z_j^+ \notin E(G)$.

Next, we show that both $L(T) \cup L_1(T)$ and $L(T) \cup L_2(T)$ are independent sets. Set $T_b = T - \{z_i z_i^+\} + \{z_i x_i\}$ for $z_i \in L_1'(T) \cup L_2'(T)$. Then $T_b$ is a spanning tree with $L(T_b) \subseteq (L(T) - \{x_i\}) \cup \{z_i^+\}$. By Remark 3.1, $L(T_b)$ is independent in $G$. Hence, $z_i^+ x_j \notin E(G)$ for $j \in [0, t] - \{i\}$. On the other hand, by Claims 3.1 and 3.5, $L(T)$ is independent in $G$ and $z_i^+ \notin N_G(x_i)$ for $z_i \in L_1'(T) \cup L_2'(T)$.

Therefore, $U$ is independent in $G$. □

**Claim 3.8.** *$\alpha(G) = k + 1 + \lceil \frac{k}{r-3} \rceil$, $|I_1| + |I_2| = t = k$, $|I_1| + \lceil \frac{|I_2|}{r-3} \rceil = \lceil \frac{k}{r-3} \rceil$, and $k = p(r-3)$ for some integer $p > 1$.*

**Proof.** Recall that $|I_1| + |I_2| = t \ge k$ and $|L_1(T)| = |L_1'(T)| = |I_1|$. By Claim 3.6, $|L_2(T)| \ge |L_2'(T)| \ge \lceil \frac{|I_2|}{r-3} \rceil$. This together with Claim 3.7 and the assumption $\alpha(G) \le k + \lceil \frac{k+1}{r-3} \rceil - \lfloor \frac{1}{|r-k-3|+1} \rfloor$, we have

$$\begin{aligned} \alpha(G) \ \ge \ |U| \quad &= |L(T)| + |L_1(T)| + |L_2(T)| \\ &\ge t + 1 + |I_1| + \left\lceil \frac{|I_2|}{r-3} \right\rceil \\ &\ge t + 1 + \left\lceil \frac{t}{r-3} \right\rceil \\ &\ge k + 1 + \left\lceil \frac{k}{r-3} \right\rceil, \end{aligned}$$

which implies $\alpha(G) = k + 1 + \lceil \frac{k}{r-3} \rceil$, $|I_1| + |I_2| = t = k$, $|I_1| + \lceil \frac{|I_2|}{r-3} \rceil = \lceil \frac{k}{r-3} \rceil$, and $k = p(r-3)$ for some integer $p > 1$. □

Recall that $L_2(T) = \{z_i^+ : z_i \in L_2'(T)\}$. By Claim 3.8, we have $p = \frac{k}{r-3}$ with $p > 1$ and there is a partition $\{X_1, \ldots, X_p\}$ of $L(T) - \{x_0\}$ satisfying that $|X_i| = 1$ for $i \in I_1$, and $|X_i| = r-3$ for $i \in I_2$. By relabeling $x_1, \ldots, x_k$ (if necessary), we may assume that for each $i \in [1, p]$, $x_i \in X_i$. For $i \in I_2$, let $X_i = \{x_{i_1}, \ldots, x_{i_{r-3}}\}$, where $i_1 = i$. Then $z_{i_1}^+ = \ldots = z_{i_{r-3}}^+ = z_i^+$. We denote $F^* = \{z_i : i \in [1, p],\ z_i \in V(T(x_0, y_0))\}$. By Claim 3.2, we assume that $y_0 \in V(T(x_0, r_{i_1}])$ for some $i_1 \in [1, k]$. Denote by $r_0$ the first branch vertex of $T$ on $T[x_0, r_{i_1}]$ (possible $r_0 = r_{i_1}$) and $r_0^+$ the successor of $r_0$ on $T[x_0, x_{i_1}]$.

**Case 1** $F^* = \emptyset$.

**Claim 3.9.** *There exists $z_0 \in V(T(x_0, y_0))$ such that $z_0 \notin N_G[x_0]$ and $V(T[z_0^+, y_0]) \subseteq N_G(x_0)$, where $z_0^+$ is the successor of $z_0$ on $T[x_0, x_{i_1}]$.*

**Proof.** Suppose that Claim 3.9 is false. Then $N_G[x_0] \cap V(T[x_0, r_{i_1}]) = V(T[x_0, y_0])$ and $r_0, r_{i_1}$ and $y_0$ are all on the path $T[x_0, x_{i_1}]$. If $y_0 \in V(T(r_0, r_{i_1}])$, then $x_0 r_0^+ \in E(G)$, a contradiction to Claim 3.4. If $y_0 \in V(T(x_0, r_0])$, since $G - y_0$ is a connected graph, there exists $u_0 \in V(T(x_0, y_0))$ satisfying that $u_0$ has a neighbour $v_0$ in $T - T[x_0, y_0]$. Set $T^* = T - \{u_0 u_0^+\} + \{x_0 u_0^+\}$. Then $T^*$ is a spanning tree with $L(T^*) = (L(T) - \{x_0\}) \cup \{u_0\}$ that satisfies $(C1)$ and $(C2)$. Noting that $d_{T^*}(u_0, y_0) = d_T(x_0, y_0)$, we have $d_{T^*}(u_0, v_0) > d_T(x_0, y_0)$ and $g(T^*, u_0) > g(T, x_0)$. On the other hand, since $F^* = \emptyset$, we have $N_G(x_j) \cap V(T(x_0, y_0)) = \emptyset$ and $g(T^*, x_j) = g(T, x_j)$ for $1 \le j \le k$. Hence $g(T^*) > g(T)$, contrary to $(C3)$. □

**Claim 3.10.** *$\{z_0\} \cup U$ is independent in $G$.*

**Proof.** Recall that $U$ is independent in $G$.

First, we prove that $\{z_0\} \cup L(T)$ is independent in $G$. We have $z_0 \notin N_G(x_0)$ by Claim 3.9. Set $T_a = T - \{z_0 z_0^+\} + \{x_0 z_0^+\}$. Then $T_a$ is a spanning tree with $L(T_a) = (L(T) - \{x_0\}) \cup \{z_0\}$. By Remark 3.1, $z_0$ is a leaf of $T_a$ and $L(T_a)$ is independent in $G$. Hence, $z_0 x_i \notin E(G)$ for $i \in [1, k]$.

Next, we show that $\{z_0\} \cup L_1(T) \cup L_2(T)$ is independent in $G$. Set $T_b = T - \{z_0 z_0^+, z_i z_i^+\} + \{x_0 z_0^+, z_i x_i\}$ for $z_i \in L_1'(T) \cup L_2'(T)$. Since $F^* = \emptyset$ and $z_i \in V(T(x_0, r_i))$, $T_b$ is a spanning tree with $L(T_b) \subseteq (L(T) - \{x_0, x_i\}) \cup \{z_0, z_i^+\}$. By Remark 3.1, both $z_0$ and $z_i^+$ are leaves of $T_b$ and $L(T_b)$ is independent. Hence, $z_0 z_i^+ \notin E(G)$ for $z_i \in L_1'(T) \cup L_2'(T)$.

Therefore, $\{z_0\} \cup U$ is independent in $G$. □

By Claim 3.10, we have $\alpha(G) \ge |\{z_0\} \cup U| \ge k + 1 + \lceil \frac{k}{r-3} \rceil + 1$, contrary to Claim 3.8. Hence Theorem 1.8 holds for Case 1.

**Case 2** $F^* \ne \emptyset$.

Choose $z_j \in V(T(x_0, y_0))$ such that $d_T(x_0, z_j)$ is as large as possible for $z_j \in F^*$. Denote the successor of $z_j$ on $T(x_0, x_{j_1})$ and $T(x_0, x_{i_1})$ by $z_j^+$ and $z_j^*$, respectively. By Claim 3.3, we have $r_{i_1 j_1} \in V(T[z_j, r_{i_1}])$.

**Claim 3.11.** *${z_j}^+, z_j^* \notin N_G(x_0)$ and there exists $u_0 \in V(T(z_j, y_0))$ (possible $u_0 = z_j^*$) satisfying that $u_0 \notin N_G(x_0)$ and $V(T[u_0^+, y_0]) \subseteq N_G(x_0)$.*

**Proof.** If $x_0z_j{}^+ \in E(G)$, then $T_a = T - \{z_jz_j^+, r_{j_1}r_{j_1}^+\} + \{x_0z_j{}^+, z_jx_{j_1}\}$ with $L(T_a) = (L(T) - \{x_0, x_{j_1}\}) \cup \{r_{j_1}^+\}$, contrary to $(C1)$. Then $z_j{}^+ \notin N_G(x_0)$. If $r_{i_1j_1} \neq z_j$, then $z_j{}^+ = z_j^*$ and thus $z_j^* \notin N_G(x_0)$. If $r_{i_1j_1} = z_j$, then $T_b = T - \{z_jz_j^*\} + \{x_0z_j{}^*\}$ with $L(T_b) = L(T) - \{x_0\}$, contrary to $(C1)$. So $z_j^* \notin N_G(x_0)$. Therefore, there exists $u_0 \in V(T(z_j, y_0))$ (possible $u_0 = z_j^*$) satisfying that $u_0 \notin N_G(x_0)$ and $V(T[u_0^+, y_0]) \subseteq N_G(x_0)$. □

Set $L^*(T) = (L(T) - \{x_{j_1}\}) \cup \{r_{j_1}^+\}$ and $U^* = L^*(T) \cup L_1(T) \cup L_2(T)$.

**Claim 3.12.** *$U^*$ is independent in $G$.*

**Proof.** Note that $U$ is independent in $G$.

First, we show that $L^*(T)$ is independent. Set $T_a = T - \{r_{j_1}r_{j_1}^+\} + \{z_jx_{j_1}\}$. Then $T_a$ is a spanning tree with $L(T_a) = (L(T) - \{x_{j_1}\}) \cup \{r_{j_1}^+\}$. By Remark 3.1, $L(T_a)$ is independent. Hence, $r_{j_1}^+x_h \notin E(G)$ for $h \in [0, k] - \{j_1\}$.

Next, we prove that $\{r_{j_1}^+\} \cup (L_1(T) \cup L_2(T) - \{z_j^+\})$ is independent in $G$. Set $T_b = T - \{r_{j_1}r_{j_1}^+, z_hz_h^+\} + \{z_jx_{j_1}, z_hx_h\}$ for $z_h \in L_1'(T) \cup L_2'(T) - \{z_j\}$. Then by Claim 3.3 and the maximality of $d_T(x_0, z_j)$, $T_b$ is a spanning tree with $L(T_b) \subseteq (L(T) - \{x_{j_1}, x_h\}) \cup \{r_{j_1}^+, z_h^+\}$. By Remark 3.1, both $r_{j_1}^+$ and $z_h^+$ are leaves of $T_b$ and $L(T_b)$ is independent in $G$. Hence, $r_{j_1}^+z_h^+ \notin E(G)$ for $z_h \in L_1'(T) \cup L_2'(T) - \{z_j\}$.

At last, we may consider that $z_j^+r_{j_1}^+ \notin E(G)$. In fact, if $z_j^+r_{j_1}^+ \in E(G)$, then $T_c = T - \{r_{j_1}r_{j_1}^+, z_jz_j^+\} + \{z_jx_{j_1}, z_j^+r_{j_1}^+\}$ with $L(T_c) = L(T) - \{x_{j_1}\}$, contrary to $(C1)$.

Therefore, $U^*$ is independent in $G$. □

**Claim 3.13.** *$r_{i_1j_1} \notin T[r_0, y_0)$.*

**Proof.** Assume that $r_{i_1j_1} \in T[r_0, y_0)$. This together with Claim 3.2 and $r_{i_1j_1} \in V(T[z_j, r_{i_1}])$ implies that $r_{i_1j_1} \in V(T[z_j, y_0))$. Let $u_0$ be the vertex in Claim 3.11, we have $V(T[u_0^+, y_0]) \subseteq N_G(x_0)$. By Claim 3.4, $u_0 \in V(T(r_{i_1j_1}, y_0))$. Then it follows that $r_{i_1j_1} \in V(T[z_j, u_0))$. Thus $u_0 \notin N_G(z_j^+)$. Otherwise, if $r_{i_1j_1} \in V(T(z_j, u_0))$, then $T' = T - \{z_jz_j^+, u_0u_0^+, r_{i_1j_1}r_{i_1j_1}^+\} + \{x_0u_0^+, u_0z_j^+, x_{j_1}z_j\}$ is a spanning tree with $L(T') \subseteq L(T) - \{x_0, x_{j_1}\} + \{r_{i_1j_1}^+\}$, contrary to $(C1)$. If $r_{i_1j_1} = z_j$, then $T'' = T - \{z_jz_j^+, u_0u_0^+\} + \{x_0u_0^+, u_0z_j^+\}$ is a spanning tree with $L(T'') \subseteq L(T) - \{x_0\}$, contrary to $(C1)$.

Now we show that $\{u_0\} \cup U^*$ or $\{r_{i_1j_1}^+\} \cup U^*$ is independent in $G$.

Note that $U^*$ is independent. Assume that $w \in \{u_0, r_{i_1j_1}^+\}$.

First, we show that $\{w\} \cup L^*(T)$ is independent. Set

$$T_a := \begin{cases} T - \left\{u_0u_0^+, r_{j_1}r_{j_1}^+\right\} + \left\{x_0u_0^+, z_jx_{j_1}\right\}, & \text{if } w = u_0; \\ T - \left\{r_{i_1j_1}r_{i_1j_1}^+, r_{j_1}r_{j_1}^+\right\} + \left\{x_0u_0^+, z_jx_{j_1}\right\}, & \text{if } w = r_{i_1j_1}^+. \end{cases}$$

Then by Claim 3.3, $T_a$ is a spanning tree with $L(T_a) \subseteq (L(T) - \{x_0, x_{j_1}\}) \cup \{w, r_{j_1}^+\}$. By Remark 3.1, both $w$ and $r_{j_1}^+$ are leaves of $T_a$ and $L(T_a)$ is independent in $G$. By Claims 3.4 and 3.11, $w \notin N_G(x_0)$. Hence, $wr_{j_1}^+ \notin E(G)$ and $wx_h \notin E(G)$ for $h \in [0, k] - \{j_1\}$.

Next, we prove that $\{u_0\} \cup L_1(T) \cup L_2(T)$ or $\{r_{i_1j_1}^+\} \cup L_1(T) \cup L_2(T)$ is independent in $G$. Note that $w \notin N_G(z_j^+)$. For $z_h \in L_1'(T) \cup L_2'(T) - \{z_j\}$, we set

$$T_b := \begin{cases} T - \left\{u_0u_0^+, z_hz_h^+\right\} + \left\{x_0u_0^+, z_hx_h\right\}, & \text{if } z_h^+ \notin F^*; \\ T - \left\{r_{i_1j_1}r_{i_1j_1}^+, z_jz_j^+, z_hz_h^+\right\} + \left\{z_h^+r_{i_1j_1}^+, z_hx_{h_1}, z_jx_{j_1}\right\}, & \text{if } z_h^+ \in F^* \text{ and } r_{i_1j_1}^+z_h^+ \in E(G). \end{cases}$$

- If $z_h^+ \notin F^*$, then by Claim 3.3, $T_b$ is a spanning tree with $L(T_b) \subseteq (L(T) - \{x_0, x_h\}) \cup \{u_0, z_h^+\}$. By Remark 3.1, both $u_0$ and $z_h^+$ are leaves of $T_b$ and $L(T_b)$ is independent. Hence, $u_0z_h^+ \notin E(G)$ for $z_h \in L_1'(T) \cup L_2'(T)$.
- If $z_h^+ \in F^*$ and $r_{i_1j_1}^+z_h^+ \in E(G)$, then by Claim 3.3, $T_b$ is a spanning tree of $G$ with $L(T_b) = (L(T) - \{x_{h_1}, x_{j_1}\}) \cup \{z_j^+\}$, contrary to $(C1)$. Thus $r_{i_1j_1}^+z_h^+ \notin E(G)$. Hence, $r_{i_1j_1}^+z_h^+ \notin E(G)$ for $z_h \in L_1'(T) \cup L_2'(T)$.

Therefore, $\{u_0\} \cup U^*$ or $\{r_{i_1j_1}^+\} \cup U^*$ is independent in $G$. Thus $\alpha(G) \geq |U^*| + 1 \geq k + 1 + \lceil \frac{k}{r-3} \rceil + 1$, contrary to Claim 3.8. This proves Claim 3.13. □

By Claim 3.13, $r_{i_1j_1} \in V(T[y_0, r_{i_1}]) \cup V(T[y_0, r_{j_1}])$. Without loss of generality, assume that $r_{i_1j_1} \in V(T[y_0, r_{j_1}])$.

**Claim 3.14.** *One of the following two statements holds.*

(i) *$u_0 \notin N_G(z_j^+)$ or there exists $w_0 \in V(T(z_j^+, u_0))$ satisfying that $w_0 \notin N_G(z_j^+)$ and $V(T[w_0^+, u_0]) \subseteq N_G(z_j^+)$;*

(ii) *$u_0 = z_j^+$ or $V(T[z_j^+, u_0]) \subseteq N_G[z_j^+]$.*

**Proof.** Suppose that Claim 3.14(ii) is false. Then $|V(T[z_j^+, u_0])| \geq 3$ and $V(T[z_j^+, u_0]) \nsubseteq N_G[z_j^+]$. If $u_0 \in N_G(z_j^+)$, then since $V(T[z_j^+, u_0]) \nsubseteq N_G[z_j^+]$, there is $w_0 \in V(T(z_j^+, u_0))$ satisfying that $w_0 \notin N_G(z_j^+)$ and $V(T[w_0^+, u_0]) \subseteq N_G(z_j^+)$. □

**Subcase 2.1** Claim 3.14*(i) holds.*

In this subcase, $w_0 \notin N_G(z_j^+) \cup N_G(x_0)$. In fact, suppose that $x_0w_0 \in E(G)$. Then by Claim 3.3, $T^* = T - \{r_{j_1}r_{j_1}^+, w_0w_0^+, z_jz_j^+\} + \{z_jx_{j_1}, x_0w_0, z_j^+w_0^+\}$ is a spanning tree with $L(T^*) = (L(T) - \{x_0, x_{j_1}\}) \cup \{r_{j_1}^+\}$, contrary to $(C1)$.

**Claim 3.15.** *If $u_0 \notin N_G(z_j^+)$, then $\{u_0\} \cup U^*$ is independent in $G$.*

**Proof.** Note that $U^*$ is independent in $G$.

First, we show that $\{u_0\} \cup L^*(T)$ is independent in $G$. Set $T_a = T - \{u_0u_0^+, r_{j_1}r_{j_1}^+\} + \{x_0u_0^+, z_jx_{j_1}\}$. Then by Claim 3.3, $T_a$ is a spanning tree of $G$ with $L(T_a) \subseteq (L(T) - \{x_0, x_{j_1}\}) \cup \{u_0, r_{j_1}^+\}$. By Remark 3.1, both $u_0$ and $r_{j_1}^+$ are leaves of $T_a$ and $L(T_a)$ is an independent set. By Claim 3.11, $u_0 \notin N_G(x_0)$. Hence, $u_0r_{j_1}^+ \notin E(G)$ and $u_0x_h \notin E(G)$ for $h \in [0, k] - \{j_1\}$.

Next, we prove that $\{u_0\} \cup L_1(T) \cup L_2(T)$ is independent in $G$. Note that $u_0 \notin N_G(z_j^+)$. For $z_h \in L_1'(T) \cup L_2'(T) - \{z_j\}$, we set

$$T_b := \begin{cases} T - \left\{u_0u_0^+, z_hz_h^+\right\} + \left\{x_0u_0^+, z_hx_{h_1}\right\}, & \text{if } z_h^+ \notin F^*; \\ T - \left\{u_0u_0^+, z_hz_h^+, r_hr_h^+\right\} + \left\{x_0u_0^+, z_hx_{h_1}, x_{j_1}z_j\right\}, & \text{if } z_h^+ \in F^*. \end{cases}$$

Then by Claim 3.3, $T_b$ is a spanning tree of $G$ with

$$L(T_b) \subseteq \begin{cases} (L(T) - \{x_0, x_h\}) \cup \left\{u_0, z_h^+\right\}, & \text{if } z_h^+ \notin F^*; \\ \left(L(T) - \left\{x_0, x_{h_1}, x_{j_1}\right\}\right) \cup \left\{u_0, z_h^+, r_h^+\right\}, & \text{if } z_h^+ \in F^*. \end{cases}$$

By Remark 3.1, both $u_0$ and $z_h^+$ are leaves of $T_b$ and $L(T_b)$ is independent in $G$. Hence, $u_0z_h^+ \notin E(G)$ for $z_h \in L_1'(T) \cup L_2'(T)$.

Therefore, $\{u_0\} \cup U^*$ is independent in $G$. □

**Claim 3.16.** *If $u_0 \in N_G(z_j^+)$, then $\{w_0\} \cup U^*$ is independent in $G$.*

**Proof.** Note that $U^*$ is independent in $G$.

First, we show that $\{w_0\} \cup L^*(T)$ is independent in $G$. Set $T_a = T - \{w_0w_0^+, z_jz_j^+, r_{j_1}r_{j_1}^+\} + \{x_0u_0^+, z_j^+w_0^+, z_jx_{j_1}\}$. Then by Claim 3.3, $T_a$ is a spanning tree of with $L(T_a) \subseteq (L(T) - \{x_0, x_{j_1}\}) \cup \{w_0, r_{j_1}^+\}$. By Remark 3.1, $w_0$ and $r_{j_1}^+$ are two leaves of $T_a$ and $L(T_a)$ is independent in $G$. If $x_0w_0 \in E(G)$, then by Claim 3.3, $T^* = T - \{r_{j_1}r_{j_1}^+, w_0w_0^+, z_jz_j^+\} + \{z_jx_{j_1}, x_0w_0, z_j^+w_0^+\}$ is a spanning tree with $L(T^*) = (L(T) - \{x_0, x_{j_1}\}) \cup \{r_{j_1}^+\}$, contrary to $(C1)$. Thus $w_0 \notin N_G(x_0)$. Hence, $w_0r_{j_1}^+ \notin E(G)$ and $w_0x_h \notin E(G)$ for $h \in [0, t] - \{j_1\}$.

Next, we prove that $\{w_0\} \cup L_1(T) \cup L_2(T)$ is independent in $G$. Note that $w_0 \notin N_G(z_j^+)$. For $z_h \in L_1'(T) \cup L_2'(T) - \{z_j\}$, we set

$$T_b := \begin{cases} T - \left\{w_0w_0^+, z_jz_j^+, z_hz_h^+\right\} + \left\{z_j^+w_0^+, z_hx_{h_1}, z_jx_{j_1}\right\}, & \text{if } z_h^+ \notin F^*; \\ T - \left\{w_0w_0^+, z_jz_j^+, z_hz_h^+, r_hr_h^+\right\} + \left\{z_j^+w_0^+, x_0u_0^+, z_hx_{h_1}, z_jx_{j_1}\right\}, & \text{if } z_h^+ \in F^*. \end{cases}$$

Then by Claim 3.3, $T_b$ is a spanning tree with

$$L(T_b) \subseteq \begin{cases} \left(L(T) - \left\{x_{h_1}, x_{j_1}\right\}\right) \cup \left\{w_0, z_h^+\right\}, & \text{if } z_h^+ \notin F^*; \\ \left(L(T) - \left\{x_0, x_{h_1}, x_{j_1}\right\}\right) \cup \left\{w_0, z_h^+, r_h^+\right\}, & \text{if } z_h^+ \in F^*. \end{cases}$$

By Remark 3.1, both $w_0$ and $z_h^+$ are leaves of $T_b$ and $L(T_b)$ is independent in $G$. Hence, $w_0z_h^+ \notin E(G)$ for $z_h \in L_1'(T) \cup L_2'(T)$.

Therefore, $\{w_0\} \cup U^*$ is independent in $G$. □

**Subcase 2.2** Claim 3.14*(ii) holds and $y_0 \neq r_{j_h}$ for some $1 \leq h \leq r - 3$.*

**Claim 3.17.** *$deg_T(x) = 2$ for any $x \in V(T[z_j^+, y_0^-])$.*

**Proof.** Suppose that $deg_T(x) \geq 3$ for some $x \in V(T[z_j^+, y_0^-])$. Denote the successor of $x$ on $T[x_0, y_0]$ by $x^+$. If $x \in V(T[u_0, y_0^-])$, then $x^+ \in N_G(x_0)$. Set $T^* = T - \{xx^+\} + \{x_0x^+\}$. Then $T^*$ is a spanning tree with $L(T^*) = L(T) - \{x_0\}$, contrary to $(C1)$. If $x \in V(T[z_j^+, u_0^-])$, then if $|V(T[z_j^+, u_0])| = 2$, then $x = z_j^+$. Set $T' = T - \{z_jz_j^+, r_{j_1}r_{j_1}^+\} + \{x_0u_0^+, z_jx_{j_1}\}$. Then $T'$ is a spanning tree with $L(T') = (L(T) - \{x_0, x_{j_1}\}) \cup \{r_{j_1}^+\}$, contrary to $(C1)$. If $|V(T[z_j^+, u_0])| \geq 3$, then set $T'' = T - \{xx^+, z_jz_j^+, r_{j_1}r_{j_1}^+\} + \{x_0u_0^+, z_jx_{j_1}, z_j^+x^+\}$. Then $T''$ is a spanning tree with $L(T'') = (L(T) - \{x_0, x_{j_1}\}) \cup \{r_{j_1}^+\}$, contrary to $(C1)$. □

By Claims 3.3 and 3.17, $V(T[x_0, r_{i_1j_1}]) \supseteq V(T[x_0, y_0])$. Denote the successor of $y_0$ on $T[x_0, x_{j_1}]$ by $y_0^+$.

**Claim 3.18.** *$\{y_0{}^+\} \cup U^*$ is independent in $G$.*

**Proof.** Note that $U^*$ is independent in $G$.

First, we show that $\{y_0{}^+\} \cup L^*(T)$ is independent in $G$. Set $T_a = T - \{y_0y_0{}^+, z_jz_j^+\} + \{x_0y_0, z_jx_{j_1}\}$. Then by Claim 3.3, $T_a$ is a spanning tree with $L(T_a) \subseteq (L(T) - \{x_0, x_{j_1}\}) \cup \{y_0^+, z_j^+\}$. By Remark 3.1, both $y_0^+$ and $z_j^+$ are two leaves of $T_a$ and $L(T_a)$ is independent in $G$. So $y_0{}^+x_h \notin E(G)$ for $h \in [1, t] - \{j_1\}$. By the choice of $y_0$, $y_0{}^+x_0 \notin E(G)$. If $y_0{}^+r_{j_1}^+ \in E(G)$, then set $T_b = T_a - \{r_{j_1}r_{j_1}^+\} + \{y_0{}^+r_{j_1}^+\}$. Thus $L(T_b) = (L(T) - \{x_0, x_{j_1}\}) \cup \{z_j^+\}$, contrary to $(C1)$.

Next, we prove that $\{y_0{}^+\} \cup L_1(T) \cup L_2(T)$ is independent. If $y_0^+ = z_h^+$ for some $z_h \in L_1'(T) \cup L_2'(T) - \{z_j\}$, then we need to consider the graph $G[y_0, x_0, y_0^-, y_0^+, x_{h_1}, \ldots, x_{h_{r-3}}]$. Since $G$ is $K_{1,r}$-free, we have $y_0^+y_0^- \in E(G)$ or $x_0^+y_0^- \in E(G)$ or $y_0^+x_{h_l} \in E(G)$ for some $1 \le l \le r-3$.

- $y_0^+y_0^- \in E(G)$. Then $T' = T - \{y_0y_0^+, y_0y_0^-, r_{h_1}r_{h_1}^+\} + \{x_0y_0, y_0^+y_0^-, y_0x_{h_1}\}$ is a spanning tree with $L(T') = (L(T) - \{x_0, x_{h_1}\}) \cup \{r_{h_1}^+\}$, contrary to $(C1)$.
- $x_0y_0^- \in E(G)$. Then $T'' = T - \{y_0y_0^-, z_jz_j^+, r_{h_1}r_{h_1}^+\} + \{x_0y_0^-, z_jx_{j_1}, y_0x_{h_1}\}$ is a spanning tree with $L(T'') = (L(T) - \{x_0, x_{j_1}, x_{h_1}\}) \cup \{z_j^+, r_{h_1}^+\}$, contrary to $(C1)$.
- $y_0^+x_{h_l} \in E(G)$ for some $1 \le l \le r-3$. Then $T''' = T - \{y_0y_0^+, z_jz_j^+, r_{h_l}r_{h_l}^+\} + \{x_0y_0, z_jx_{j_l}, y_0^+x_{h_l}\}$ is a spanning tree with $L(T''') = (L(T) - \{x_0, x_{j_l}, x_{h_l}\}) \cup \{z_j^+, r_{h_l}^+\}$, contrary to $(C1)$.

So $y_0^+ \ne z_h^+$ for any $z_h \in L_1'(T) \cup L_2'(T) - \{z_j\}$. Set $T_c = T_a - \{z_hz_h^+\} + \{z_hx_{h_1}\}$ for $z_h \in L_1'(T) \cup L_2'(T) - \{z_j\}$. Then $T_c$ is a spanning tree with $L(T_c) \subseteq (L(T) - \{x_0, x_{j_1}, x_h\}) \cup \{y_0^+, z_j^+, z_h^+\}$. By Remark 3.1, $y_0^+, z_j^+$ and $z_h^+$ are leaves of $T_c$ and $L(T_c)$ is independent in $G$. Hence, $y_0^+z_h^+ \notin E(G)$ for $z_h \in L_1'(T) \cup L_2'(T)$.

Therefore, $\{y_0{}^+\} \cup U^*$ is independent in $G$. □

**Subcase 2.3** Claim 3.14*(ii) holds and* $y_0 = r_{j_h}$ *for any* $1 \le h \le r-3$.

**Claim 3.19.** $\{r_{j_1}^+, \ldots, r_{j_{r-3}}^+, x_0, y_0^-\}$ *is an independent set and* $deg_T(y_0) = r-2$.

**Proof.** We first show that $\{r_{j_1}^+, \ldots, r_{j_{r-3}}^+, x_0\}$ is independent in $G$. By Claim 3.4, $\{r_{j_s}^+, x_0\}$ is an independent set for $1 \le s \le r-3$. If $r-3=1$, then $\{r_{i_1}^+, x_0\}$ is independent in $G$. If $r-3 \ge 2$, set $T^* = T - \{y_0r_{j_p}^+, y_0r_{j_q}^+\} + \{z_jx_{j_p}, z_jx_{j_q}\}$ for $1 \le p \ne q \le r-3$. Then by Claim 3.3, $T^*$ is a spanning tree with $L(T^*) = (L(T) - \{x_{j_p}, x_{j_q}\}) \cup \{r_{j_p}^+, r_{j_q}^+\}$. By Remark 3.1, $L(T^*)$ is independent in $G$. Hence, $r_{j_p}^+r_{j_q}^+ \notin E(G)$.

Next, if $x_0y_0^- \in E(G)$, then by Claim 3.3, $T' = T - \{y_0^-y_0, z_jz_j^+\} + \{x_0y_0^-, z_jx_{j_1}\}$ is a spanning tree with $L(T') = (L(T) - \{x_0, x_{j_1}\}) \cup \{z_j^+\}$, contrary to $(C1)$; if $y_0^-r_{j_s}^+ \in E(G)$ for some $1 \le s \le r-3$, then by Claim 3.3, $T'' = T - \{y_0^-y_0, y_0r_{j_s}^+\} + \{x_0y_0, r_{j_s}^+y_0^-\}$ is a spanning tree with $L(T'') = (L(T) - \{x_0, x_{j_s}\}) \cup \{r_{j_s}^+\}$, contrary to $(C1)$. Therefore, $\{r_{j_1}^+, \ldots, r_{j_{r-3}}^+, x_0, y_0^-\}$ is independent in $G$.

Now we prove that $deg_T(y_0) = r-2$. Assume that $deg_T(y_0) \ge r-1$ and $y_0x \in E(T)$ for $x \notin \{r_{j_1}^+, \ldots, r_{j_{r-3}}^+, y_0^-\}$. Since $G$ is $K_{1,r}$-free and $\{y_0, r_{j_1}^+, \ldots, r_{j_{r-3}}^+, x_0, y_0^-\}$ is an induced $K_{1,r-1}$, we have $xy \in E(G)$ for some $y \in \{r_{j_1}^+, \ldots, r_{j_{r-3}}^+, x_0, y_0^-\}$. By Claim 3.4, $x_0x \notin E(G)$. If $xr_{j_s}^+ \in E(G)$ for some $1 \le s \le r-3$, then $T_a = T - \{y_0r_{j_s}^+, y_0x\} + \{xr_{j_s}^+, z_jx_{j_s}\}$ is a spanning tree with $L(T_a) = L(T) - \{x_{j_s}\}$, contrary to $(C1)$; if $xy_0^- \in E(G)$, then $T_b = T - \{y_0^-y_0, y_0x\} + \{x_0y_0, xy_0^-\}$ is a spanning tree with $L(T_b) = L(T) - \{x_0\}$, contrary to $(C1)$. □

By Claim 3.19, we have $deg_T(y_0) = r-2$. Let $T_f$ be a connected component $T - z_j^+$ such that $z_j \in V(T_f)$. Then by Claim 3.17, $B(T) - \{y_0\} = B(T_f)$. Denote $B^* = B(T_f) - \{z_j\}$. Then $T^* = T - \{y_0r_{j_1}^+, \ldots, y_0r_{j_{r-3}}^+, z_jz_j^+\} + \{x_{j_1}z_j, \ldots, x_{j_{r-3}}z_j, x_0y_0\}$ is a spanning tree with $|L(T^*)| = |L(T)|$. Assume that $d_T(x_0, z_j) = a$ and $d_T(z_j^+, y_0) = b$. Note that $deg_{T^*}(z) = deg_T(z)$, $d_{T^*}(z_j^+, z) = d_T(x_0, z) + b + 1$ for any $z \in B^*$ and $deg_{T^*}(y_0) = 2$, $deg_T(y_0) = r-2$, $d_{T^*}(z_j^+, y_0) = b$, $d_T(x_0, y_0) = a+b+1$ and $deg_{T^*}(z_j) = deg_T(z_j) + r - 4$, $d_{T^*}(z_j^+, z_j) = a+b+1$. Hence,

$$
\begin{aligned}
& f(T^*, z_j^+) - f(T, x_0) = \sum_{z \in I(T^*)} (deg_{T^*}(z) - 2) d_{T^*}(z_j^+, z) - \sum_{z \in I(T)} (deg_T(z) - 2) d_T(x_0, z) \\
& = \{ \sum_{z \in B^*} (deg_{T^*}(z) - 2) d_{T^*}(z_j^+, z) + \sum_{z \in \{z_j, y_0\}} (deg_{T^*}(z) - 2) d_{T^*}(z_j^+, z) \} \\
& - \{ \sum_{z \in B^*} (deg_T(z) - 2) d_T(x_0, z) + \sum_{z \in \{z_j, y_0\}} (deg_T(z) - 2) d_T(x_0, z) \} \\
& = \sum_{z \in B^*} (deg_T(z) - 2)(b+1) + (deg_T(z_j) + r - 4 - 2)(a+b+1) \\
& - \{ (r-2-2)(a+b+1) + (deg_T(z_j) - 2)a \} \\
& = \sum_{z \in B^*} (deg_T(z) - 2)(b+1) + (deg_T(z_j) - 2)(b+1) \\
& = \sum_{z \in B^* \cup \{z_j\}} (deg_T(z) - 2)(b+1)
\end{aligned}
$$

This together with $b+1 > 0$ and $(C2)$ implies that $\sum_{z \in B^* \cup \{z_j\}} (deg_T(z) - 2) \le 0$. Thus $deg_T(z) = 2$ for $z \in B^* \cup \{z_j\}$. By Claim 3.17, we have $B(T) = \{y_0\}$. In this subcase, $k = r-3$ and thus, $p = \frac{k}{r-3} = 1$, contrary to Claim 3.8.

By Claims 3.15, 3.16 and 3.18, we have $\alpha(G) \ge |U^*| + 1 \ge k + 1 + \lceil \frac{k}{r-3} \rceil + 1$, contrary to Claim 3.8. This completes the proof of Case 2.

## 4. Proof of Theorem 1.11

We define $(G_1, G_2, x)$ a *separation* of a connected graph $G$ if $G$ can be decomposed into two nonempty connected subgraphs $G_1$ and $G_2$ with $V(G_1) \cap V(G_2) = \{x\}$. We call a path $P$ an $x$-path if $P$ has an end vertex $x$. An $(x, Y)$-path is a path starting at $x$ and ending at a vertex of $Y$, where the internal vertices are not in $\{x\} \cup Y$. An $(x, Y, t)$-fan is a set of $t$ internally disjoint $(x, Y)$-paths with distinct terminal vertices in $Y$.

**Lemma 4.1.** *Let $G$ be a connected $K_{1,4}$-free graph and $(G_1, G_2, x)$ be a separation of $G$. If $G_i$ is a block and $\alpha(G_i) \leq 3$, then $G_i$ has a Hamiltonian $x$-path for $i = 1, 2$.*

**Proof.** For convenience, we can only take $G_1$ into consideration. Assume that $G_1$ has no Hamiltonian $x$-path. Choose an $x$-path $P$ in $G_1$ such that

$(C4)$ $P$ is as long as possible.

Suppose that $x$ and $y$ are the end vertices of $P$. Obviously, $N_{G_1}(y) \subseteq V(P)$ as $(C4)$ and $G_1$ has no $x$-claw as $G$ being $K_{1,4}$-free. We set a direction from $x$ to $y$ in $P$. Since $P$ is not a hamiltonian $x$-path and $G_1$ is 2-connected, there exists a $(z, P, 2)$-fan such that $zQ_1u_1$ and $zQ_2u_2$ are two disjoint $(z, P)$ paths, where $z \in V(G_1 - P)$ and $u_1, u_2 \in V(P)$. Let $y_0$ be a neighbour of $y$ in $G_1$ such that $d_P(y, y_0) = \max_{v \in N_{G_1}(y)} d_P(y, v)$. Obviously, $y \neq u_2$.

By the choice of $(C4)$ and $y_0$, it is easy for us to check the following claim. □

**Claim 4.2.**

(1) $d(u_1, u_2) \geq 2$;
(2) $\{z, u_1{}^+, u_2{}^+\}$ *and* $\{z, u_1{}^+, y\}$ *are two independent sets;*
(3) *if* $u_1{}^-$ *exists, then* $\{z, u_1{}^-, u_2{}^-\}$ *is also an independent set;*
(4) $u_1{}^-, u_1{}^+, u_2{}^- \notin N_{G_1}(y)$.

Next, we will consider two assumptions:

We first assume that $x = u_1$. By Claim 4.2, $\{x^+, z, y\}$ is independent. Since $G_1$ has no $x$-claw, we have $x \notin N_{G_1}(y)$. Note that $y_0 \neq x^+, u_2{}^-$ and $\delta(G) \geq 2$.

If $y_0 \in V(x^{++}Pu_2{}^{--})$, then $\{y_0^+, z, x^+, u_2{}^+\}$ is an independent set. In fact, we set

$$P' = \begin{cases} xPy_0y \overleftarrow{P} y_0^+ z & \text{if } zy_0^+ \in E(G_1); \\ xQ_1zQ_2u_2Pyy_0 \overleftarrow{P} x^+y_0^+Pu_2{}^- & \text{if } x^+y_0^+ \in E(G_1); \\ xPy_0y \overleftarrow{P} u_2^+y_0^+Pu_2Q_2z & \text{if } u_2{}^+y_0^+ \in E(G_1). \end{cases}$$

Then $P'$ is an $x$-path in $G_1$ with $|V(P')| > |V(P)|$, which contradicts $(C4)$. By Claim 4.2(1), $\{z, x^+, u_2{}^+\}$ is independent. Hence, $\{y_0^+, z, x^+, u_2{}^+\}$ is independent in $G_1$, a contradiction to $\alpha(G_1) \leq 3$.

If $y_0 \in V(u_2Py)$, then we can utilize the similar discussion to Claim 3.5 in Theorem 1.8 to find $z_0 \in V(y_0Py)$ such that $z \in N_{G_1}(y)$ for all $z \in V(y_0Pz_0)$ and $yz_0{}^+ \notin E(G_1)$. Set

$$P'' = \begin{cases} xQ_1zQ_2u_2Pz_0y \overleftarrow{P} z_0{}^+x^+Pu_2{}^- & \text{if } x^+z_0{}^+ \in E(G_1); \\ xPz_0y \overleftarrow{P} z_0{}^+z & \text{if } zz_0{}^+ \in E(G_1). \end{cases}$$

Then $P''$ is an $x$-path in $G_1$ with $|V(P'')| > |V(P)|$, which contradicts $(C4)$. Note that $\{x^+, z, y\}$ is independent. Hence, $\{x^+, z, z_0{}^+, y\}$ is independent in $G_1$, a contradiction to $\alpha(G_1) \leq 3$.

We now assume that $x \neq u_1$. By Claim 4.2(4), $u_1{}^-, u_1{}^+, u_2{}^- \notin N_{G_1}(y)$.

If $y_0 \in V(xPu_1{}^{--})$, then $\{y_0^+, z, u_1{}^+, u_2{}^+\}$ is an independent set. In fact, if $y_0^+u_1{}^+ \in E(G_1)$, then $P' = xPy_0y\overleftarrow{P}u_1{}^+y_0^+Pu_1Q_1z$ is an $x$-path in $G_1$, which contradicts $(C4)$. By the similar discussion as above, we have $y_0^+u_2{}^+, y_0^+z \notin E(G_1)$. Note that $\{z, u_1{}^+, u_2{}^+\}$ is an independent set by Claim 4.2(1). Hence, $\{y_0^+, z, u_1{}^+, u_2{}^+\}$ is an independent set, a contradiction to $\alpha(G_1) \leq 3$.

If $y_0 \in V(u_1Pu_2{}^{--})$, then we can easily see that $\{y_0^+, z, u_1{}^-, u_2{}^+\}$ is an independent set, a contradiction to $\alpha(G_1) \leq 3$; if $y_0 \in V(u_2Py)$, then it is easy to check that $\{y_0^+, z, u_1{}^-, u_2{}^-\}$ is an independent set, a contradiction to $\alpha(G_1) \leq 3$.

Hence, $G_1$ has a Hamiltonian $x$-path. With the similar argument in $G_1$, $G_2$ also has a Hamiltonian $x$-path. Then Lemma 4.1 holds.

**Proof of Theorem 1.11..** If $G$ is 2-connected, then the result holds by Corollary 1.10. If $G$ is not 2-connected, suppose that $\alpha(B) \geq 3$ for every block $B$ in $G$ and $G$ is a minimal counterexample to Theorem 1.11. Let $x$ be a cut vertex in $G$ and $(G_1, G_2, x)$ be a separation of $G$. Obviously, $\alpha(G_1) + \alpha(G_2) - 1 \leq \alpha(G) \leq \alpha(G_1) + \alpha(G_2)$ and $\alpha(G_i) \geq 3$.

**Case 1** *$\alpha(G_1) > 5$ and $\alpha(G_2) > 5$.*

Let $k_i$ be an integer such that $k_i = \lfloor \frac{\alpha(G_i)-4}{2} \rfloor$ for $i = 1, 2$. Then $k_i \geq 1$.

On one hand, $2k + 5 \geq \alpha(G) \geq \alpha(G_1) + \alpha(G_2) - 1 \geq 2(k_1 + k_2 + 1) + 5$. Hence, $k_1 + k_2 + 1 \leq k$. $G_i$ satisfies the condition in Theorem 1.11 and the independence number of every block in $G_i$ is also no less than 3. On the other hand, since $G$

is a minimal counterexample to Theorem 1.11, $G_i$ has a spanning tree with at most $k_i$ branch vertices. Then $|B(T_1 \cup T_2)| \le |B(T_1) \cup B(T_2) \cup \{x\}| \le |B(T_1)| + |B(T_2)| + 1 \le k_1 + k_2 + 1$.

Hence, $T_1 \cup T_2$ is a spanning tree of $G$ with at most $k$ branch vertices, a contradiction with $G$ being a counterexample.

**Case 2** $\alpha(G_1) > 5$ *and* $3 \le \alpha(G_2) \le 5$.

Let $k_1$ be an integer such that $k_1 = \lfloor \frac{\alpha(G_1)-4}{2} \rfloor$ and $k_2 = 0$. Then $k_1 \ge 1$ and $\alpha(G_2) \le 5 = 2k_2 + 5$.

On one hand, $2k + 5 \ge \alpha(G) \ge \alpha(G_1) + \alpha(G_2) - 1 \ge 2k_1 + 4 + 3 - 1$. Hence, $k_1 + 1 \le k$. $G_i$ satisfies the condition in Theorem 1.11 and the independence number of every block in $G_i$ is also no less than 3. On the other hand, since $G$ is a minimal counterexample to Theorem 1.11, $G_i$ has a spanning tree with at most $k_i$ branch vertices. Then $|B(T_1 \cup T_2)| \le |B(T_1) \cup B(T_2) \cup \{x\}| \le |B(T_1)| + |B(T_2)| + 1 \le k_1 + k_2 + 1 = k_1 + 1$.

Therefore, $T_1 \cup T_2$ is a spanning tree of $G$ with at most $k$ branch vertices, a contradiction with $G$ being a counterexample.

**Case 3** $3 \le \alpha(G_1) \le 5$ *and* $3 \le \alpha(G_2) \le 5$.

Let $k_i = 0$. Then $\alpha(G_i) \le 5 = 2k_i + 5$.

On one hand, $\alpha(G) \ge \alpha(G_1) + \alpha(G_2) - 1 \ge 5$. $G_i$ satisfies the condition in Theorem 1.11 and the independence number of every block in $G_i$ is also no less than 3. On the other hand, since $G$ is a minimal counterexample to Theorem 1.11, $G_i$ has a spanning tree with at most $k_i$ branch vertices. Then $|B(T_1 \cup T_2)| \le |B(T_1) \cup B(T_2) \cup \{x\}| \le |B(T_1)| + |B(T_2)| + 1 \le k_1 + k_2 + 1 = 1$. In fact, $\alpha(G) \le 5$. Otherwise, $2k + 5 \ge \alpha(G) \ge 6$. That is, $k \ge 1 \ge |B(T_1 \cup T_2)|$. Then $T_1 \cup T_2$ is a spanning tree of $G$ with at most $k$ branch vertices, a contradiction with $G$ being a counterexample.

Therefore, $\alpha(G) = 5$ and $\alpha(G_1) = \alpha(G_2) = 3$. By Lemma 4.1, $G_i$ has a Hamiltonian $x$-path $P_i$ for $i = 1, 2$. Then $P_1 \cup P_2$ is a Hamiltonian path in $G$, a contradiction with $G$ being a counterexample. Hence Theorem 1.11 holds. □

## Data availability

No data was used for the research described in the article.

## Acknowledgments

We are grateful to the anonymous referees for their valuable comments and suggestions of this paper.

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